Vectors

AS Exam Questions

A few basic principles are used over and again in M1 exam questions. Hopefully the solutions given below should give a few ideas as to what might appear. Examiner Reports usually state that vectors is the worst answered question on the paper, but with a little thought half marks is easily attainable.

Example 9

A particle B moves with constant acceleration $(3\mathbf{i} + 7\mathbf{j})\text{ms}^{-2}$. At time t its velocity is v ms^{-1}. When t = 0, v = $(12\mathbf{i} - 14\mathbf{j})\text{ms}^{-1}$.

a) find the time when B is moving parallel to the vector **i**.

b) find the speed of B when t = 8.

c) find the angle between the direction of motion of B and the vector **i** when t = 8.

a) The particle will be moving parallel to the vector **i** when it has no **j** component.

The velocity at any time is given by:

Velocity = initial velocity + (acceleration × time)

$$= (12\mathbf{i} - 14\mathbf{j}) + t(3\mathbf{i} + 7\mathbf{j}) \qquad (1)$$

We are only interested in the **j** component and particularly when it is zero:

$$-14\mathbf{j} + 7t\mathbf{j} = 0$$

Therefore: $t = 2$

b) By substituting t = 8 into equation (1) we can find the velocity of the particle. Speed is the magnitude of the velocity.

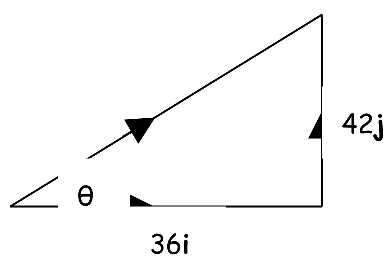
$$\text{Velocity} = (12\mathbf{i} - 14\mathbf{j}) + t(3\mathbf{i} + 7\mathbf{j})$$

When t = 8 $\quad \text{Velocity} = (12\mathbf{i} - 14\mathbf{j}) + 8 \times (3\mathbf{i} + 7\mathbf{j})$

$$= (36\mathbf{i} + 42\mathbf{j})\ \text{ms}^{-1}$$

$$\text{Speed} = \sqrt{36^2 + 42^2} = 55.3\text{ms}^{-1}$$

c) The velocity at any time gives the direction of motion.



Therefore the direction is given by:

$$\theta = \text{Tan}^{-1}\left(\frac{42}{36}\right)$$

$$\theta = 49.4^{\circ}$$

Example 10

A particle has position vector (3**i** + 7**j**) and is moving with speed 39ms^{-1} in the direction (12**i** – 5**j**). Find the position vector at time t = 5 and the distance from the point (3,4) at this time.

The hint in the question is that the velocity of the particle must be a multiple of (12i – 5j)ms^{-1}. The magnitude of this vector is 13, but the speed of the particle is 39ms^{-1}, therefore the velocity of the particle must be (36i – 15j)ms^{-1}. This one little trick is often used but if you don't spot it the question is almost impossible.

The position vector of the particle at any time is given by:

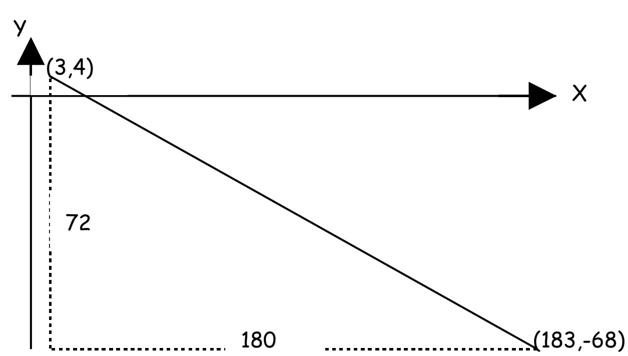
Position = initial position + (velocity × time)

r = (3**i** + 7**j**) + t(36**i** – 15**j**)

When t = 5

r = (183**i** – 68**j**)

The distance between the point and the particle can be found by Pythagoras theorem.



Distance = $\sqrt{72^2 + 180^2}$ = 193.9m

The final two examples are of the more challenging type. Setting the equations for the position vectors is pretty straight forward as long as you follow the rule:

$$r = \text{initial position} + (\text{velocity} \times \text{time})$$

In most cases the ships or aeroplanes are on a collision course and therefore you must set the equations equal to each other and equate coefficients. These questions may carry in excess of 10 marks but you should be able to make a start.

Example 11

A command post O monitors the movement of two of its ships in the Gulf. At 1200 hrs a battleship (B) has position $(-2\mathbf{i} + 10\mathbf{j})$ km relative to O and has constant velocity of $(3\mathbf{i} + 2\mathbf{j})$ kmh^{-1}. A frigate (F) is at the point with position vector $(4\mathbf{i} + 5\mathbf{j})$ km and has constant velocity $(-3\mathbf{i} + 7\mathbf{j})$ kmh^{-1}, where **i** and **j** are unit vectors directed due east and due north respectively.

a) The captain of one ship has been taken ill, show that the two ships will collide.

The command post contacts the battleship and orders it to reduce its speed to move with velocity $(2\mathbf{i} + 2\mathbf{j})$ kmh^{-1}.

b) Find an expression for the vector *BF* at time t hours after noon.

c) Find the distance between B and F at 1400 hrs.

d) Find the time at which F will be due north of B.

a) The position of each ship is given by it's position vector:

$$\text{position} = \text{initial position} + (\text{velocity} \times \text{time})$$

So for the battleship:

$$r_b = (-2\mathbf{i} + 10\mathbf{j}) + t(3\mathbf{i} + 2\mathbf{j})$$

And for the frigate:

$$r_f = (4\mathbf{i} + 5\mathbf{j}) + t(-3\mathbf{i} + 7\mathbf{j})$$

If the two ships are to collide then for some value of t their respective **i** and **j** components must be equal.

Therefore by equating **i**'s:

$$-2 + 3t = 4 - 3t$$

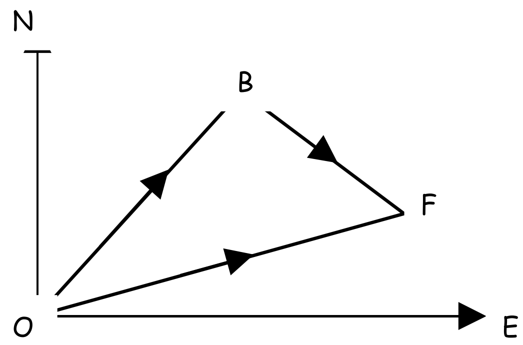
$$t = 1$$

Substituting the value of $t = 1$ into r_b and r_b gives the same position vector of ($\mathbf{i} + 12\mathbf{j}$). Therefore the two ships will collide after one hour at the point with position vector ($\mathbf{i} + 12\mathbf{j}$).

b) The position vector for the battleship must change to take account of the new velocity:

$$r_b = (-2\mathbf{i} + 10\mathbf{j}) + t(2\mathbf{i} + 2\mathbf{j})$$

We have been asked to find the vector *BF* as shown in the diagram below.



By triangle law:

$$\vec{OB} + \vec{BF} = \vec{OF}$$

$$\vec{BF} = \vec{OF} - \vec{OB}$$

Where $\vec{OB} = r_b$ and $\vec{OF} = r_f$

Therefore: $\vec{BF} = r_f - r_b$

$= (4\mathbf{i} + 5\mathbf{j}) + t(-3\mathbf{i} + 7\mathbf{j}) - ((-2\mathbf{i} + 10\mathbf{j}) + t(2\mathbf{i} + 2\mathbf{j}))$

$\vec{BF} = (6\mathbf{i} - 5\mathbf{j}) + t(-5\mathbf{i} + 5\mathbf{j})$

c) The magnitude of $\vec{BF}$ gives the distance between the two ships, at 1400 hrs, t = 2

$= (6\mathbf{i} - 5\mathbf{j}) + 2(-5\mathbf{i} + 5\mathbf{j})$

$= (-4\mathbf{i} + 5\mathbf{j})$

Magnitude = $\sqrt{41} Km$

d) If F is due north of B, then *BF* will have no **i** component.

$$BF = (6\mathbf{i} - 5\mathbf{j}) + t(-5\mathbf{i} + 5\mathbf{j})$$

$$6\mathbf{i} - 5t\mathbf{i} = 0$$

t = 1hr 12mins

Example 12

Two cars P and Q are moving on straight horizontal roads with constant velocities. The velocity of P is $25ms^{-1}$ due east and the velocity of Q is $(10\mathbf{i} + 8\mathbf{j})ms^{-1}$. Initially P is at rest at the origin and Q has the position vector 230**i**m relative to the origin. At time t seconds the position vectors of P and Q are *r* metres and *s* metres respectively.

a) Find expressions for *r* and *s* in terms of t.
b) Write an expression for PQ.
c) Find the time when the bearing of Q from P is 045°
d) Find the time when the cars are 200m apart.

a) The position vector for car P is given by:

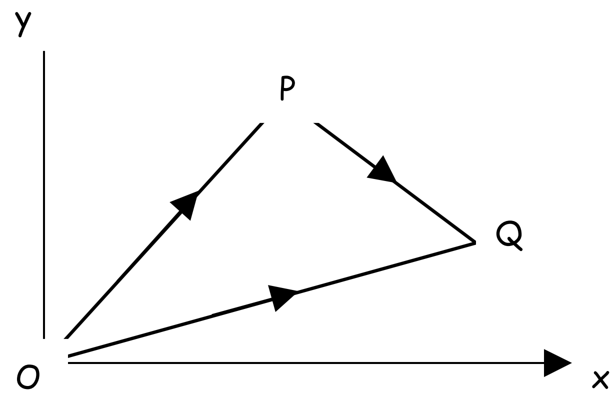
r = initial position + (velocity × time)

$$r = 0 + 25t\mathbf{i}$$

Therefore for Q:

$$s = 230\mathbf{i} + t(10\mathbf{i} + 8\mathbf{j})$$

b) Using the triangle law:



$OP + PQ = OQ$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

Where $OP = \mathbf{r}$ and $OQ = \mathbf{s}$

Therefore: $PQ = \mathbf{s} - \mathbf{r}$

$= 230\mathbf{i} + t(10\mathbf{i} + 8\mathbf{j}) - 25t\mathbf{i}$

$= 230\mathbf{i} + t(8\mathbf{j} - 15\mathbf{i})$

c) If the bearing of Q from P is 045°, then the vector PQ must be parallel to the vector $(\mathbf{i} + \mathbf{j})$.

Therefore

$$230\mathbf{i} + t(8\mathbf{j} - 15\mathbf{i}) = m(\mathbf{i} + \mathbf{j})$$

Equating coefficients:

$230 - 15t = m$ (1) $8t = m$ (2)

Substituting gives:

$$230 - 15t = 8t$$

$$t = 10$$

The bearing of Q from P is 045° after 10 seconds.

d) The magnitude of PQ will give the distance that the two cars are apart. We need the value of t where $|PQ| = 200$
Rewriting PQ gives:

$$PQ = (230 - 15t)\mathbf{i} + 8t\mathbf{j}$$

$$|PQ| = \sqrt{(230 - 15t)^2 + 64t^2}$$

$$|\overrightarrow{PQ}| = \sqrt{52900 - 6900t - 225t^2 + 64t^2}$$

$$200^2 = 52900 - 6900t - 161t^2$$

$$161t^2 + 6900t - 12900 = 0$$

$$t = 1.79$$

<u>Example 13</u>

At 1200 hrs the position vectors of two helicopters A and B are r_A and r_B as outlined below. Use the velocity vectors v_A and v_B to give the position vectors of A and B at a time t hours after noon.

$r_A = (3\mathbf{i} + 5\mathbf{j})$ $r_B = (5\mathbf{i} + 2\mathbf{j})$ $v_A = (\mathbf{i} + 2\mathbf{j})$ $v_B = (2\mathbf{i} +$
$3\mathbf{j})$

Find an expression at time t hours after noon for the position vector of B relative to A.

If d is the distance in Km between the two helicopters find the value of d^2 in terms of t.

Find the time at which the helicopters are closest together. Give the value of the minimum distance.

a) For Helicopters A and B the position vectors at time t hours after noon are:

$r_A = (3\mathbf{i} + 5\mathbf{j}) + t(\mathbf{i} + 2\mathbf{j})$ $r_B = (5\mathbf{i} + 2\mathbf{j}) + t(2\mathbf{i} + 3\mathbf{j})$

b) The position vector of B relative to A is given by:

$$r_B - r_A = 2\mathbf{i} - 3\mathbf{j} + t(\mathbf{i} + \mathbf{j})$$

$$= (2 + t)\mathbf{i} + (t - 3)\mathbf{j}$$

c) The magnitude of the vector in part (b) gives the distance between the two particles.
Therefore:

$$d^2 = (2 + t)^2 + (t - 3)^2$$

$$d^2 = 2t^2 - 2t + 13$$

By completing the square we can find the minimum value:

$$d^2 = 2[t^2 - t + 6.5]$$

Remembering to halve the t coefficient:

$(-0.5)^2 = 0.25$ we need 6.5

$$d^2 = 2[(t - 0.5)^2 + 6.25]$$

The minimum value occurs when t = 0.5. Therefore the minimum distance between the two helicopters is $\sqrt{12.5}$ Km